Section 5.3
- Use the change of base formula to evaluate logarithms
- Use the properties of logarithms (Product, Quotient, and Power Properties) to condense or expand a logarithmic expression

Section 5.4
- Solve simple and more complicated logarithmic or exponential equations

Section 5.5
- Recognize common exponential and logarithmic models
- Use the common exponential and logarithmic models to solve real world problems

Section 6.1
- Draw an angle in standard position (degrees or radians)
- Identify the quadrant and angle lies in (degrees or radians)
- Find a co-terminal angle and describe what it means for two angles to be co-terminal (degrees or radians)
- Find complementary and supplementary angles (degrees or radians)
- Convert from degrees to radians and from radians to degrees
- Describe what a radian is
- Find the arc length or sector area of a circle

Section 6.2
- Solve for missing angles or sides in a right triangle using right triangle trigonometry
- Evaluate the trigonometric functions at 30, 45, and 60 degrees and \( \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3} \) radians
- Identify and be able to use the fundamental trigonometric identities (reciprocal, quotient)

Section 6.3
- Evaluate the six trigonometric functions given a point on the terminal ray of the angle
- Evaluate the six trigonometric functions given a trigonometric value and information regarding the angle’s quadrant
- Evaluate functions of quadrant angles
- Find reference angle in both radians and degrees
- Evaluate a trigonometric function using reference angles in both radians and degrees without using your calculator

Section 6.4
- Graph the sine and cosine curves.
- Find and use the amplitude and period to graph the sine or cosine function.
- Find and use the vertical and horizontal translations to graph the sine or cosine function.

Section 6.5
- Graph tangent, cotangent, cosecant, and secant curves.
- Find the asymptotes of each function and use them to graph the function.
5.3 Properties of Logarithms

**Change-of-Base Formula:**
- Base 10: $\log_a x = \frac{\log x}{\log a}$
- Base e: $\log_a x = \frac{\ln x}{\ln a}$
- Base b: $\log_a x = \frac{\log_b x}{\log_b a}$

Use the change of base to rewrite the logarithm in given base.

1. Rewrite $\log_{11} 4$ into base 10.
2. Rewrite $\log_2 2$ into base e.
3. Rewrite $\log_4 17$ in base 12

**Properties of Logarithms (With log base a and the natural logarithm)**

1. **Product Property:** $\log_a (uv) = \log_a u + \log_a v$ or $\ln(uv) = \ln u + \ln v$
2. **Quotient Property:** $\log_a \frac{u}{v} = \log_a u - \log_a v$ or $\ln \frac{u}{v} = \ln u - \ln v$
3. **Power Property:** $\log_a u^n = n \log_a u$ or $\ln u^n = n \ln u$

**Using Properties of Logarithms**

Write the logarithm in terms of $\ln 3$ and $\ln 2$.

4. $\ln 6$
5. $\ln \left( \frac{2}{27} \right)$
6. $\ln 12$

**Expanding Logarithmic Expressions**

7. $\ln \left( \frac{x^5 z}{3 p^2} \right)$
8. $\log_3 x^2 \sqrt[4]{\frac{y^4}{z}}$
Condensing Logarithmic Expressions

9. $13 \log_8 x - \log_8 g - 4 \log_8 z + \log_8 f$

10. $\frac{1}{3} (2 \ln(x + 5) - 6 \ln x - \ln 4)$

11. $3 \log_2 x + 4 \log_2 2 + 7 \log_3 y - 5 \log_2 z$
5.4 Exponential and Logarithmic Equations

Rationale: Student will use their knowledge of the properties of logarithmic and exponential functions to begin solving logarithmic and exponential equations.

Performance Objective: Students will be able to solve logarithmic and exponential equations.

Strategies for Solving Exponential Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One properties of exponential or logarithmic functions.

2. Rewrite the exponential equation in logarithmic form and apply the inverse property of logarithmic functions.

Solving Exponential Equations

1. \(3^x = 81\)
2. \(2(7^x) = 98\)

3. \(4(6^x) = 80\)
4. \(2e^x + 9 = 17\)

5. \(6(7^{x-5}) + 8 = 56\)
6. \(e^{5x-6} = e^x\)

7. \(e^{2x} - 5e^x - 14 = 0\)

Application

8. Determine the amount of time it would take $7000 to double in an account that pays 0.75% interest, compounded continuously. (Use \(A = Pe^{rt}\).)
Strategies for Solving Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One properties of exponential or logarithmic functions.

2. Rewrite the logarithmic equation in exponential form and apply the inverse property of exponential functions.

Solving Logarithmic Equations

9. \( \log_y(2x - 7) = \log_y(-4x + 19) \)

10. \( \ln x = 15 \)

11. \( \ln(4x) - \ln(x + 9) = 0 \)

12. \( \log_3 x = 4 \)

13. \( 3\ln x - 16 = 23 \)

14. \( 6\log_2 8x = 36 \)

15. \( \log(x + 1) + \log(x + 4) = 1 \)

16. \( \log_5 5x - \log_5(x - 1) = 2 \)

Application

17. The retail sales \( y \) (in billions) of e-commerce companies in the United States from 2002 to 2007 can be modeled by \( y = -549 + 236.7 \ln t \), \( 12 \leq t \leq 17 \) where \( t \) represents the year, with \( t=12 \) corresponding to 2002. During which year did the sales reach $275 billion?
5.5 Exponential and Logarithmic Models

Common Exponential and Logarithmic Models

1. Exponential Growth Model: \( y = ae^{bx}, \quad b > 0 \)
2. Exponential Decay Model: \( y = ae^{-bx}, \quad b > 0 \)
3. Gaussian Model: \( y = ae^{-(x-b)^2} \)
4. Logistic Growth Model: \( y = \frac{a}{1 + be^{-rx}} \)
5. Logarithmic Models: \( y = a + b \ln x \) and \( y = a + b \log x \)

Exponential Growth

1. The function \( L(t) = A(1 - e^{-kt}) \) can be used to measure the amount \( L \) learned at time \( t \). The number \( A \) represents the amount to be learned, and the number \( k \) measures the rate of learning. Suppose a student has an amount \( A \) of 200 vocabulary words to learn. A psychologist determines that the student learned 20 vocabulary words after 5 minutes. Determine the rate of learning \( k \).

2. A medical study suggests that the risk \( R \) (as a percentage) of having a car accident with alcohol in your blood can be described by \( R = 1.5e^{ka} \), where \( a \) the concentration of alcohol in the blood and \( k \) is a constant. Suppose a concentration of alcohol in the blood of 0.11 results in a 10% risk (\( R = 10 \)) of an accident. Write the equation representing risk as a function of blood alcohol and use it to answer the following.
   a) What is the risk if the concentration is 0.17?
   
   b) If anyone whose risk of having an accident is 6% or more should not drive, what concentration of alcohol in the blood should be used to test a person’s ability to continue driving?
Exponential Decay

3. The population of a midwestern city follows the exponential law. If the population decreased from 900,000 to 800,000 from 2011 to 2013, what will the population be in 2015?

Gaussian Models

This a model commonly used in probability and statistics because it represents populations that are normally distributed. The shape is often referred to as the bell curve.

For standard normal distributions, the model takes the form \[ y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \]

Where the maximum y value of the bell curve appears is the average value of the population. The x-value that corresponds to the maximum y value represents the average value of the independent variable.

4. IQ scores roughly follow the model

\[ y = 0.027 e^{-\frac{(x-130)^2}{450}}, \quad 60 \leq x \leq 200, \]

where x is the IQ score. Estimate the average IQ score.
Logistic Growth Models

The logistic growth model shows a rapid growth rate and then a declining rate of growth. It is also called a sigmoidal curve.

5. On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread is modeled by $y = \frac{5000}{1 + 4999e^{-0.8t}}$, $t \geq 0$ where $y$ is the total number of students infected after $t$ days. The college will cancel classes when 30% or more of the students are infected.

   a) How many students are infected after 4 days?
   
   b) After how many days will the college cancel class?

Logarithmic Models

The Richter scale is one way of converting seismographic readings into numbers that provide an easy reference for measuring the magnitude $M$ of an earthquake. All earthquakes are compared to a so-called zero-level earthquake whose seismographic reading measures .001 millimeter at a distance of 100 kilometers from the epicenter. An earthquake whose seismographic reading measures $x$ millimeters has a magnitude $M(x)$ given by $M(x) = \log \left( \frac{x}{x_0} \right)$ where $x_0 = 10^{-3}$ is the reading of a zero-level earthquake the same distance from its epicenter.

   a) What is the magnitude of an earthquake whose seismographic reading is .1 millimeter at a distance of 100 kilometers from its epicenter? (simplify completely)

   b) Find the magnitude of an earthquake whose seismographic reading is 10 millimeters at a distance of 100 kilometers from its epicenter. (simplify completely)
6.1 Angles and Their Measures

Angles

Angle – represents a rotation about a point. ($\theta$)

Standard Position – vertex at the origin, initial ray along the positive x-axis.

- Clockwise is a negative rotation.

- Counter clockwise is a positive rotation

Radian – One radian is the measure of a central angle $\theta$ that intercepts an arc $s$ equal in length to the radius $r$ of the circle. This means $\theta = \frac{s}{r}$ where $\theta$ is measured in radians. (Note that $\theta = 1$ when $s = r$)

*The circumference of the circle is $2\pi r$, so it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of $s = 2\pi r$. 
An angle can be written in either degrees or radians.

1 degree = ______ radians  
1 radian = ______ degrees

**Converting degrees to radians:** Multiply the angle by ______, then simplify.

1. Convert from degrees to radians.
   
   a) $690^\circ$  
   b) $-140^\circ$
Converting radians to degrees: Multiply the angle by \( \frac{180}{\pi} \), then simplify.

2. Convert from radians to degrees.

   a) \( \frac{5\pi}{4} \)
   b) \( \frac{17\pi}{6} \)
   c) \( -\frac{7\pi}{6} \)
   d) 2 radians

Coterminal Angles: Two angles, when in standard position, have the same terminal angle.

3. Find two co-terminal angles for the given angle.

   A) \( 115^\circ \)  
   B) \( \frac{\pi}{4} \)  
   C) \( -\frac{3\pi}{2} \)  
   D) \( 440^\circ \)

Complementary and Supplementary Angles

Complementary Angles – The sum of their measures is 90° in degrees or \( \frac{\pi}{2} \) in radians.
*Complements are positive angles!

Supplementary Angles – The sum of their measures is 180° in degrees or \( \pi \) in radians.

4. Find the measure of the angle complementary to:

   a) \( 63^\circ \)  
   b) \( \frac{\pi}{5} \)  
   c) \( 146^\circ \)

5. Find the measure of the angle supplementary to:

   a) \( 135^\circ \)  
   b) \( \frac{2\pi}{3} \)  
   c) \( 1.9 \) radians

Applications

**Sector** – A region bounded by a central angle and the intercepted arc. (It is like a slice of pie.)

*The area of a sector is a fraction of the whole area.

*The length of an arc is a fraction of the whole circumference.
**Radian Formulas:** To be used when your angle is in radians.

Arc length | Sector area
---|---

6. A sprinkler on a golf course fairway sprays water over a distance of 90 feet and rotates through an angle of 150°. Find the area of the fairway watered by the sprinkler.

7. A birthday cake has a diameter of 16 inches. You cut the cake into 6 pieces. What is the sector area of one piece of cake?

**Linear and Angular Speeds**

Linear speed \( v = \frac{\text{arc length}}{\text{time}} = \frac{s}{t} \) (measures how fast the particle moves at a constant speed along a circular path)

Angular speed \( \omega = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t} \) (measures how fast the angle changes)

8. The second hand of clock is 10.2 centimeters long. Find the linear speed of the tip of this second hand as it passes around the clock face.

9. The blades of a wind turbine are 116 feet long. The propeller rotates at 15 revolutions per minute.
   a) Find the angular speed of the propeller in radians per minute.
   b) Find the linear speed of the tips of the blades.
6.2 Right Triangle Trigonometry

Let \( \theta \) be defined as the acute angle of a right triangle.

- The sine function is defined as \( \sin \theta = \frac{\text{opp}}{\text{hyp}} \).

- The cosine function is defined as \( \cos \theta = \frac{\text{adj}}{\text{hyp}} \).

- The tangent function is defined as \( \tan \theta = \frac{\text{opp}}{\text{adj}} \).

- The cosecant function is defined as \( \csc \theta = \frac{\text{hyp}}{\text{opp}} \).

- The secant function is defined as \( \sec \theta = \frac{\text{hyp}}{\text{adj}} \).

- The cotangent function is defined as \( \cot \theta = \frac{\text{adj}}{\text{opp}} \).

1. Find the six trigonometric values of \( \theta \), using the triangle to the right.

2. Find the other five trigonometric values of \( \theta \), given that \( \tan \theta = \frac{6}{7} \) and that \( \theta \) is in the 3\textsuperscript{rd} quadrant.

Solving the triangle (Applications)

1. The safety instructions for a 20 ft. ladder indicate that the ladder should not be inclined at more than a 65° angle with the ground. Suppose the ladder is leaned against a house at this angle. Find the distance from the base of the house to the foot of the ladder and the height reached by the ladder.
2. The new Ferris wheel in Las Vegas, NV is 550 feet tall. You are standing at a point on the ground some distance in feet from the center of the Ferris wheel’s base watching your friend ride the Ferris wheel. When they reach the very top, the measure of the angle of elevation from where you are standing is 75°. How far are you from the center of the Ferris wheel’s base?

Evaluating Trigonometric Functions of 45°

Evaluating Trigonometric Functions of 30° and 60°
3. A historic lighthouse is 300 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 600 yards long. Find the acute angle $\theta$ between the bike path and the walkway.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\theta \text{ (degrees)} & 0^\circ & 30^\circ & 45^\circ & 60^\circ & 90^\circ \\
\hline
\theta \text{ (radians)} & 0 & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} & \frac{\pi}{2} \\
\hline
\sin \theta & 0 & \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} & 1 \\
\hline
\cos \theta & 1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\
\hline
\tan \theta & 0 & \frac{1}{\sqrt{3}} & 1 & \frac{\sqrt{3}}{1} & \text{Undefined} \\
\hline
\end{array}
$$
6.3 Trigonometric Functions of Any Angle

Definitions

P(x, y) is the point on the circle $x^2 + y^2 = r^2$. $\theta$ is the angle in standard position with terminal ray $\overrightarrow{OP}$.

- The sine function is defined as $\sin \theta = \frac{y}{r}$.
- The cosine function is defined as $\cos \theta = \frac{x}{r}$.
- The tangent function is defined as $\tan \theta = \frac{y}{x}$.
- The cosecant function is defined as $\csc \theta = \frac{r}{y}$.
- The secant function is defined as $\sec \theta = \frac{r}{x}$.
- The cotangent function is defined as $\cot \theta = \frac{x}{y}$.

Evaluating Trigonometric Functions

1. If the terminal ray of an angle in standard position passes through the point (9, -12). Find all six trig values.

2. Given $\tan \theta = -\frac{5}{12}$ and $\cos \theta < 0$, find $\cos \theta$, $\csc \theta$, and $\cot \theta$.
3. Find:
   
   a) \( \sin 540^\circ \)       b) \( \sin \frac{-\pi}{2} \)
   
   c) \( \sin(-180^\circ) \)   d) \( \cos 90^\circ \)
   
   e) \( \cos 270^\circ \)   f) \( \tan \pi \)

4. Is \( \sin 320^\circ \) positive or negative?  

5. Is \( \cos \frac{3\pi}{4} \) positive or negative?
6. Is \( \sin 2.41 \) positive or negative?  

7. Is \( \tan \frac{\pi}{6} \) positive or negative?  

8. If \( \cos \theta = -1 \), what are some possible values for \( \theta \)?  

**Reference Angles**  
Reference Angle – For any angle (\( \theta \)), the acute positive angle (\( \theta' \)) formed by the terminal ray of \( \theta \) and the x-axis.  

Express the following angle in terms of a reference angle.  

9. \( \sin 210^\circ \)  

10. \( \cos 660^\circ \)  

11. \( \cos (1.7) \)  

12. \( \cos \frac{7\pi}{6} \)  

13. \( \cos (-121)^\circ \)  

**Evaluating Trigonometric Functions Using Reference Angles**  
14. \( \cos \frac{5\pi}{4} \)  

15. \( \sin \frac{7\pi}{3} \)  

16. \( \sec (-330^\circ) \)  

17. \( \csc (120^\circ) \)  

18. \( \tan \left(\frac{\pi}{6}\right) \)  

19. \( \cot 135^\circ \)
6.4 Graphs of Sine and Cosine Functions

Basic Sine Curve \( y = \sin x \)  
Basic Cosine Curve \( y = \cos x \)

**Amplitude and Period**

General equations for sine and cosine:

\[
\begin{align*}
y &= d + a \sin(bx-c) \\
y &= d + a \cos(bx-c)
\end{align*}
\]

Amplitude represents half the distance between the maximum and minimum values of the function and is given by

\[
\text{amplitude} = |a|.
\]

The amplitude stretches or shrinks the graph vertically.

If the \( a \) is negative, the amplitude is positive, but the graph will be reflected over the \( x \)-axis.

1. Graph the following using amplitude.
   
   a) \( y = 3 \cos x \)
   
   b) \( y = -\frac{1}{2} \sin x \)

2. Graph the following using period.
   
   a) \( y = \cos \left( \frac{x}{3} \right) \)
   
   b) \( y = \sin(2x) \)
Translations of Sine and Cosine Curves

Horizontal Translation
The constant $c$ in the general equations for sine and cosine creates a horizontal translation (shift). The graph of $y = d + \sin(bx+c)$ or $y = d + \cos(bx+c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$. By solving for $x$, you can find the interval for one cycle to be $\frac{c}{b} \leq x \leq \frac{c}{b} + \frac{2\pi}{b}$, where $\frac{c}{b}$ is the phase shift and $\frac{2\pi}{b}$ is the period.

3. Sketch the graph.
   a) $y = \sin(x + \frac{\pi}{4})$
   b) $y = \cos(x - \pi)$

Vertical Translation
The constant $d$ in the general equations for sine and cosine create a vertical translation (shift). When $d > 0$, the graph shifts $d$ units upward. When $d < 0$, the graph shifts $d$ units downward. The graph oscillates about the line $y = d$.

4. Sketch the graph.
   a) $y = -5 + \sin x$
   b) $y = 4 + \cos x$

Find the amplitude, the period, the phase shift, and the key points, then graph the function.

a) $y = -3 + 2\sin(3x - \frac{\pi}{4})$

b) $y = \frac{3}{2} - \sin(2x + \frac{\pi}{2})$
6.5 Graphs of Other Trigonometric Functions

Cosecant
cscx = (1/sinx) and the Period is $2\pi$.
Vertical asymptotes where sinx is zero, which occurs at $x = n\pi$.
Graph by first graphing sinx and then take the reciprocals of the y values.

1. Sketch a graph of:
   a) $y = \csc x$
   b) $y = -3\csc(x - \frac{\pi}{6})$.

Secant
secx = (1/cosx)
Period is $2\pi$ and vertical asymptotes where cosx is zero, which occurs at $x = \frac{\pi}{2} + n\pi$.
Graph by first graphing cosx and then take the reciprocals of the y values.

2. Sketch a graph of:
   a) $y = \sec x$
   b) $y = 1 + \sec(2x - \pi)$.
The Tangent Function
The graph of the tangent function is symmetric about the origin, it is an odd function.

\[ \tan x = \frac{\sin x}{\cos x} \], which makes tangent undefined for values at which \( \cos x = 0 \). (That is \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \).) When the graph is undefined at a value of \( x \) it has a vertical asymptote at that point.

The period of the tangent function is \( \pi \), so there are other vertical asymptotes when \( x = \frac{\pi}{2} + n\pi \).

The general equation for tangent is \( y = \tan(bx-c) \).

Two consecutive asymptotes can be found by solving the equations \( bx - c = -\frac{\pi}{2} \) and \( bx - c = \frac{\pi}{2} \).

The midpoint of these asymptotes is the \( x \)-intercept.

Sketch a graph of:

a) \( y = \tan x \)

b) \( y = 2\tan \left( \frac{x}{4} \right) \)

The Cotangent Function
The cotangent function also has a period of \( \pi \).

From the identity \( \cot x = \frac{\cos x}{\sin x} \), we know that there are vertical asymptotes where \( \sin x \) is zero, which occurs at \( x = n\pi \).

Two consecutive vertical asymptotes can be found by solving the equations \( bx - c = 0 \) and \( bx - c = \pi \).

3. Sketch a graph of:

a) \( y = \cot x \)

b) \( y = -3\cot(2x) \).