SQUEEZING POLYNOMIAL ROOTS
A NONUNIFORM DISTANCE

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Abstract. Given a polynomial with all real roots, the Polynomial Root Squeezing Theorem states that moving two roots an equal distance toward each other, without passing other roots, will cause each critical point to move toward \((r_i + r_j)/2\), or remain fixed. In this note, we extend the Polynomial Root Squeezing Theorem to the case where two roots are squeezed together a nonuniform distance.

1. Introduction

Given a polynomial \(p(x)\), all of whose roots are real, the Polynomial Root Dragging Theorem [1, 4] states that moving one or more roots of the polynomial to the right will cause every critical point to move to the right, or stay fixed. Moreover, no critical point moves as far as the root that is moved the farthest. But what happens to the position of a critical point when some of the roots are dragged in opposing directions?

The Polynomial Root Squeezing Theorem [2, 3] begins the analysis of this problem. Let \(r_1 \leq \cdots \leq r_n\) be the \(n\) real roots of \(p(x)\) with \(r_i \neq r_j\) interior roots. We say that a critical point is stubborn if it is a repeated root of \(p(x)\), and ordinary otherwise. Then the assertion of the Polynomial Root Squeezing Theorem is that if \(r_i\) and \(r_j\) move equal distances toward each other, without passing other roots, then each stubborn critical point which is not located at \(r_i\) or \(r_j\) will stay fixed, and each ordinary critical point moves toward \((r_i + r_j)/2\). If \(r_i\) or \(r_j\) is a repeated root of multiplicity greater than two, one of the repeated critical points will move toward \((r_i + r_j)/2\), while the others will remain fixed. In this case, the moving root which is closest to a given critical point has the most pull on that critical point. Unfortunately, this intuition does not allow us to see what happens when two distinct roots are squeezed together a nonuniform distance.

Throughout the paper we will let \(p(x)\) be a polynomial of degree \(n\) with \(n\) real roots \(r_1 \leq r_2 \leq \cdots \leq r_n\) and critical points \(c_1 \leq c_2 \leq \cdots \leq c_{n-1}\). Consider two distinct roots \(r_i < r_j\) and \(c_k\) any ordinary critical point. If we drag \(r_i\) to the right, the Polynomial Root Dragging Theorem tells us that \(c_k\) will also move to the right. If we then drag \(r_j\) to the left, the critical
point \( c_k \) moves back to the left. But how far must we drag \( r_j \) to the left in order for \( c_k \) to return to its original position? We call this distance the threshold value. For moving \( r_j \) back any smaller distance will leave \( c_k \) to the right of its original position, and moving \( r_j \) any larger distance leaves \( c_k \) to the left of its original position. In what follows we present a formula for the threshold value.

2. Squeezing Roots a Nonuniform Distance

The threshold value determines exactly what happens to the position of \( c_k \) when two roots are squeezed together a nonuniform distance.

**Theorem 2.1** (Threshold Value). Let \( p(x) \) be a monic polynomial of degree \( n \) with \( r_i < r_j, \ d \leq r_{i+1} - r_i, \) and \( 0 < h < d \). Let \( c_k \) be any ordinary critical point,

\[
\tilde{p}(x) = (x - r_i - h)(x - r_j + h + \alpha_k)q(x)
\]

with

\[
q(x) = \prod_{k \neq i, j} (x - r_k)
\]

and

\[
\alpha_k = \frac{-h(r_j - (r_i + h))q'(c_k)}{q(c_k) + (c_k - (r_i + h))q'(c_k)}
\] (2.1)

Then \( \tilde{p}'(c_k) = 0 \).

We let \( d \leq \min\{r_{i+1} - r_i, r_j - r_i\} \), so that we study the case where two roots are squeezed together without passing other roots. However, when finding the threshold value, \( r_j \) may have to pass other roots. In fact, it may even pass through \( r_i + h \). Let’s consider such an example. If \( p(x) = x(x - 1)(x - 4) \) and we drag \( r_2 = 1 \) to the right \( \frac{1}{2} \) units, the threshold value for \( c_1 \), when we drag \( r_3 = 4 \) to the left, is 2.691569405. That is, \( p(x) \) and \( \tilde{p}(x) = x(x - 1.5)(x - 1.308430595) \) have the same first critical point where we have moved \( r_2 = 1 \) to \( r_2 + h = 1.5 \) and \( r_3 = 4 \) to \( r_3 - (h + \alpha_k) = 1.308430595 \).

The hypothesis that \( c_k \) is an ordinary critical point does not need to be weakened to include stubborn critical points. If \( c_k \) is a stubborn critical point it does not move in response to \( r_i \) and \( r_j \) being squeezed. In this case \( q(c_k) = q'(c_k) = 0 \) and equation (2.1) is undefined, as it should be, since any value of \( \alpha_k \) will leave the critical point \( c_k \) fixed. However, when \( c_k \) is an ordinary critical point the value of \( \alpha_k \) is unique.

**Proof.** Let \( r_i < r_j \) be two distinct roots of \( p \) and \( c_k \) any ordinary critical point. Since

\[
p'(x) = (x - r_i + x - r_j)q(x) + (x - r_i)(x - r_j)q'(x)
\]
and
\[ p'(x) = (x - r_i + x - r_j + \alpha_k)q(x) + (x^2 - (r_i + r_j)x + r_rr_j + h(r_j - r_i - h) - \alpha_k(h + r_i - x))q'(x), \]
it follows that
\[ p'(x) - p'(x) = \alpha_k q(x) + (h(r_j - r_i - h) - \alpha_k(h + r_i - x))q'(x). \]
As \( c_k \) is a critical point of \( p(x) \),
\[ \tilde{p}'(c_k) = \alpha_k q(c_k) + (h(r_j - (r_i + h)) - \alpha_k(h + r_i - c_k))q'(c_k). \]
Setting \( \tilde{p}'(c_k) = 0 \) yields
\[ (q(c_k) + (c_k - (r_i + h))q'(c_k)) \alpha_k = -h(r_j - (r_i + h))q'(c_k). \]
In order to solve for \( \alpha_k \), we must show that
\[ q(c_k) + (c_k - (r_i + h))q'(c_k) \]
is not zero for \( h \) satisfying \( 0 < h < r_j - r_i \). By shifting \( p(x) \) we can assume that \( c_k = 0 \), so we need to show that
\[ q(0) - (r_i + h)q'(0) \]
is not zero for \( 0 < h < r_j - r_i \). If \( q'(0) = 0 \), then (since \( c_k \) is an ordinary critical point) \( q(0) \neq 0 \) and (2.3) is not zero. Therefore, we can assume that \( q'(0) \neq 0 \) in what follows.
By differentiating \( p(x) = (x - r_i)(x - r_j)q(x) \) at \( x = 0 \), we get
\[ (r_i + r_j)q(0) = r_i r_j q'(0). \]
Multiplying (2.3) by \( r_i + r_j \) and substituting the value of \( (r_i + r_j)q(0) \) from the last equation, it suffices to show
\[ r_i r_j q'(0) - (r_i + r_j)(r_i + h)q'(0) \neq 0. \]
Since \( q'(0) \neq 0 \), we can factor out \(-q'(0) \) and show that
\[ -r_i r_j + (r_i + r_j)(r_i + h) = r_i^2 + h(r_i + r_j) \]
is not zero in the range \( 0 < h < r_j - r_i \). At \( h = 0 \) the expression’s value is \( r_i^2 \) and at \( h = r_j - r_i \) its value is \( r_i^2 \). Because \( r_i < r_j \), at least one of \( r_i^2 \) and \( r_i^2 \) is positive, and the other is non-negative. Because the expression is a linear function of \( h \), \( r_i^2 + h(r_i + r_j) \) and hence \( q(c_k) + (c_k - (r_i + h))q'(c_k) \)
is non-zero throughout the range \( 0 < h < r_j - r_i \). Therefore,
\[ \alpha_k = \frac{-h(r_j - (r_i + h))q'(c_k)}{q(c_k) + (c_k - (r_i + h))q'(c_k)} \]
is defined, and for this value of \( \alpha_k \), \( \tilde{p}'(c_k) = 0. \]
\[ \square \]
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The cases where $r_i = c_k$ or $r_j = c_k$ are trivial. As intuition suggests, if $r_i = c_k$, then

$$h + \alpha_k = h + \frac{-h(r_j - (r_i + h))q'(c_k)}{(r_i - (r_i + h))q'(c_k)} = h + \frac{-h(r_j - (r_i + h))}{-h} = r_j - r_i. $$

Likewise, if $r_j = c_k$, then $h + \alpha_k = 0$.

The Polynomial Root Dragging Theorem suggests that $0 \leq h + \alpha_k \leq r_j - r_i$. This is in fact true.

**Lemma 2.2.** Under the hypothesis of Theorem 2.1,

$$h + \alpha_k = \frac{hr_j^2}{r_i - h(r_i + r_j)}$$

with $0 \leq h + \alpha_k \leq r_j - r_i$.

**Proof.** By shifting $p(x)$, we can assume that $c_k = 0$. (More generally, one can show that $c_k = 0$).

Therefore,

$$h + \alpha_k = \frac{h(r_j - c_k)^2}{(r_i - c_k)^2 + h(r_i + r_j - 2c_k)}.$$ 

However, the starting point for this formula is to consider an ordinary critical point $c_k$. So for simplicity we shift the polynomial making $c_k = 0$.)

If $r_i, r_j \neq 0$, equation (2.4) implies that $q(0) = 0$. In this case

$$h + \alpha_k = \frac{hr_j^2}{r_i - h(r_i + r_j)}. $$

If $r_i, r_j = 0$, then either $r_i = 0$ or $r_j = 0$. If $r_i = 0$ (since $r_i \neq r_j$, $r_j \neq 0$), Equation (2.4) implies that $q(0) = 0$. In this case

$$h + \alpha_k = \frac{hr_j q'(0)}{hq'(0)} = r_j.$$
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If \( r_j = 0 \), similar work shows that
\[
\frac{h + \alpha_k}{r_i^2 + h(r_i + r_j)} = 0.
\]
So when \( r_i r_j = 0 \), the formula
\[
h + \alpha_k = \frac{hr^2_j}{r_i^2 + h(r_i + r_j)}
\]
still holds.

Since,
\[
\frac{d}{dh} \left( \frac{hr^2_j}{r_i^2 + h(r_i + r_j)} \right) = \frac{r^2_j r^2_i}{(r^2_i + h(r_i + r_j))^2} \geq 0
\]
and \( 0 < h < r_j - r_i \), we have that
\[
0 \leq h + \alpha_k \leq \frac{(r_j - r_i)r^2_j}{r_i^2 + (r_j - r_i)(r_i + r_j)} = r_j - r_i.
\]

\[\square\]

We now are ready to show that \( c_k \) is the \( k \)th critical point of \( \tilde{p}(x) \).

**Theorem 2.3.** Under the hypothesis of Theorem 2.1, denote \( \tilde{r}_i = r_i + h \), \( \tilde{r}_j = r_j - h - \alpha_k \) and the critical points of \( \tilde{p}(x) \) by \( \tilde{c}_1 \leq \tilde{c}_2 \leq \cdots \leq \tilde{c}_{n-1} \). Then \( c_k = \tilde{c}_k \).

**Proof.** We will show that \( c_k \) is the \( k \)th critical point of \( \tilde{p}(x) \). Since \( 0 \leq h + \alpha_k \leq r_j - r_i \), it follows that if \( c_k \leq r_i \) or \( r_j \leq c_k \), then neither root crosses \( c_k \). Therefore \( c_k \) will be the \( k \)th critical point of \( \tilde{p}(x) \).

When \( r_i < c_k < r_j \), we show that there are exactly three possibilities:
\[
\tilde{r}_i < c_k < \tilde{r}_j,
\]
\[
\tilde{r}_j < c_k < \tilde{r}_i,
\]
\[
\tilde{r}_i = c_k = \tilde{r}_j.
\]
By shifting \( p(x) \), we can assume \( c_k = 0 \), so that \( r_i < 0 < r_j \). We first show that if \( h = |r_i| = -r_i \), then \( h + \alpha_k < r_j \). The proof of Lemma 2.2 implies that
\[
h + \alpha_k = \frac{hr^2_j}{r_i^2 + h(r_i + r_j)}
\]
is a nondecreasing function of \( h \).
Since \( 0 < h < -r_i \),
\[
h + \alpha_k < \frac{-r_i r^2_j}{r_i^2 - r_i(r_i + r_j)} = r_j.
\]
Therefore, if \( \tilde{r}_i < c_k \), then \( c_k < \tilde{r}_j \). A similar argument shows that if \( \tilde{r}_i > c_k \), then \( c_k < \tilde{r}_j \).
We now show that if $h = -r_i$, then $h + \alpha_k = r_j$. In this case,
\begin{align*}
  h + \alpha_k &= h + \frac{-h(r_j - (r_i + h))q'(0)}{q(0) - (r_i + h)q'(0)} \\
  &= -r_i + \frac{r_i r_j q'(0)}{q(0)} \\
  &= -r_i q(0) + (r_i + r_j)q(0) \\
  &= r_j.
\end{align*}

Therefore, if $\tilde{r}_i = c_k$, then $c_k = \tilde{r}_j$.

When $r_i < c_k < r_j$, it is now easy to see that $c_k$ will be the $k$th critical point of $\tilde{p}(x)$, we simply count the number of roots to the left of $c_k$. However, in each case, since $r_i$ and $r_j$ are the only moved roots, it is clear that $c_k$ is the $k$th critical point of $\tilde{p}(x)$.

In general, it seems unrealistic to know what happens to a given critical point, when squeezing roots a nonuniform distance, without accounting for the distances between the critical point and each moving root which is done in Lemma 2.2. It remains an open question to find similar estimates when more than two roots are moved in opposing directions (a uniform or nonuniform distance). This could prompt some interesting undergraduate research.

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References


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