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Sensitivity analysis of the transient torque viscosity measurement method

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Abstract
While a new type of oscillating-cup viscometer which uses a rotating magnetic field has been developed and tested, the uncertainty of such measurements has been considered to only a cursory level. This paper reports work in which the sensitivities of this new method's output are studied for the measurement of mercury with a particular device. It is concluded from the analysis that while the new method requires dramatically shorter measurement times than previous methods, the resultant uncertainty seems to be nearly the same as the classical oscillating cup method. As such, since the results are just as precise as other oscillating-cup methods, the new method should be employed for its speed advantage, allowing the viscosity of materials as functions of temperature or pressure to be economically determined.

Keywords: viscosity, sensitivity, oscillating-cup

(Some figures may appear in colour only in the online journal)
in differential form is given in equation (1), where \( \rho \) is the fluid density, \( V_\phi \) is the tangential velocity, \( r \) is the radial coordinate as measured from the axis of the viscometer and \( \nu \) is the kinematic viscosity.

\[
\rho \frac{dV_\phi}{dt} = \rho f_\phi + \rho \nu \left( V^2 V_\phi - \frac{V_\phi}{r^2} \right)
\]  

(1)

The body force \( f_\phi \) is a function of the magnetic field strength \( B_0 \), the field rotation rate \( \omega \), and the fluid electrical conductivity \( \sigma \), as defined in equations (2) and (3) as demonstrated in the literature, which assume that the magnetic field fully penetrates the sample and that the magnetic Reynolds number is low [8–11].

\[
f_\phi = \frac{f_0}{\omega R} (\omega r - V_\phi)
\]

(2)

\[
f_0 = \frac{\sigma}{2} B_0^2 \omega R.
\]

(3)

Since the viscosity is of primary interest, the electrical conductivity and magnetic field properties are all represented by the body force constant term \( f_0 \). The sample fluid is assumed to start at rest, giving the initial condition of equation (4).

\[
V_\phi(r, 0) = 0.
\]

(4)

The viscometer’s motion can be modeled by a simple rotational spring-mass-damper system, where the equation of motion is given in (5).

\[
I \frac{d^2 \alpha}{dr^2} + C \frac{d\alpha}{dr} + k\alpha - \tau = 0.
\]

(5)

In this equation, \( I \) is the rotating body’s moment of inertia (excluding the fluid), while \( k \) is the quartz thread’s spring constant and \( C \) is its hysteretic damping constant. The angular displacement \( \alpha \) is the dependent variable of the ordinary differential equation. The angular displacement is defined to start at zero, as defined in equation (6).

\[
\alpha(0) = 0.
\]

(6)

The final term of the equation, \( \tau \), includes all other moments on the rotational system and includes the forces from the fluid on the viscometer. The fluid velocity and angular rotation rate must match at the interface between the two at the vessel radius \( R \), as defined by equation (7).

\[
V_\phi(R, t) = R \frac{d\alpha}{dt}
\]

(7)

Given these conditions, the fluid-induced moment on the viscometer system can be computed by equation (8) from geometry and the shear stress.

\[
\tau = -2\pi R^2 \nu \frac{dV_\phi}{dr} \bigg|_{r=R}
\]

(8)

3. Numerics

A variety of numerical methods were employed in the examination of experimental uncertainties for the new viscometer operation mode; a detailed consideration of each method follows in the next sections.

3.1. System simulation

The equations of motion (1) and (5) must be solved numerically during the curve fit, since no analytical solutions are known for this system. The numerical solution uses the control volume finite element method (CVFEM) to discretize the equations in space, while a Crank–Nicholson method is used for time integration. This results in a method that is formally second order in space and time, and also very stable from the implicit component of the time integration.

Because CVFEM solvers are not commonly encountered, a brief overview of the method is provided here. As the name of the method suggests, CVFEM solvers result from the combination of ideas from traditional finite volume (FVM) and finite element methods (FEM) [12, 13]. A FEM-style mesh and shape functions are used to discretize the domain and provide an interpolated solution between grid points. This allows the method to be used as complex geometries, or to provide clustered nodes near important features. Using this discretization control volumes (often called cells) are defined around the nodes in the mesh, and using the divergence theorem a conservation law can be enforced for each volume; this technique is sometimes called a dual-mesh, since the same domain is broken up into both elements and cells. A diagram of the elements and control volumes used in this solver is found in figure 2.

To enforce a conservation law, traditional FVM methods are used for each cell. The conservation equation is integrated
over space and time, the divergence theorem is applied, and approximations are then used for most terms in the resulting expression. The results of the first two steps for the present problem are listed in equation (9). CVFEM solvers differ from more traditional FVM solvers in the last step by using the interpolation from the finite elements to provide the approximations for terms instead of local finite difference methods. The use of finite differences in FVM solvers has led some to call them finite difference methods (FDM), or more correctly, CVFDM; the latter name should be preferred to distinguish them from both CVFEM solvers and simpler FDM solvers.

\[
\left[ \int_V \rho V_\phi \, dV \right]_{\text{cell}} = \int_\Omega \int_V \rho_\phi \, dV \, dt + \int_\Omega \int_V \mu \, V_\phi \, dS \, dt - \int_\Omega \int_V \frac{V_\phi}{r^2} \, dV \, dt
\]

(9)

For a more general solver it is desirable to move all the terms in equation (9) to one side and then call the expression the residual for the cell, as shown in equation (10). When the residual is zero, the conservation law holds for the cell and the governing equation is satisfied in an integral sense. To solve the system at each time step, the residuals for all cells are forced to zero.

\[
R (V_\phi) = \left[ \int_V \rho V_\phi \, dV \right]_{\text{cell}} - \int_\Omega \int_V \rho_\phi \, dV \, dt - \int_\Omega \int_V \mu \, V_\phi \, dS \, dt + \int_\Omega \int_V \frac{V_\phi}{r^2} \, dV \, dt
\]

(10)

A novel method is used to incorporate the solid body rotation into the simulation; a boundary element was created which accounts for the momentum transport between the fluid and body, thus coupling the two systems through the same momentum transport mechanics as in the fluid solver. This element references the current ampoule displacement angle to compute the reactive force of the spring. The angle itself is simply the time integration of the body’s angular velocity. The residual contribution from the solid body mechanics is shown in equation (11), where \( M \) is the moment of inertia converted into an effective mass at the outer radius of the ampoule.

\[
\begin{aligned}
R (V_\phi) &= M \frac{dV_\phi}{dt} + CV_\phi + k\alpha \\
&= \int_\Delta t M \frac{dV_\phi}{dt} \, dt + \int_\Delta t (CV_\phi + k\alpha) \, dt \\
&= \left[ MV_\phi \Delta t \right] + \int_\Delta t (CV_\phi + k\alpha) \, dt
\end{aligned}
\]

(11)

A multivariate Newton’s method is used to solve the equations at each step, although only a single iteration is required since the problem is linear. This method was chosen because the mathematics of the solver are straightforward and allow for added complexity if it becomes needed. The method comes from the Taylor’s series expansion of the residual vector for all degrees of freedom in the problem, as seen in (12). By forcing the perturbed residual to zero and recognizing the residual’s derivative as the Jacobian, the final method is found in equation (13).

\[
\begin{aligned}
R (V_\phi + \delta) &= R (V_\phi) + \frac{\partial R}{\partial V_\phi} (V_\phi) \delta + \text{H.O.T.}
\end{aligned}
\]

(12)

\[
\begin{aligned}
J (V_\phi) &= \frac{\partial R}{\partial V_\phi} (V_\phi) \\
R (V_\phi + \delta) &\rightarrow 0
\end{aligned}
\]

(13)

Evaluation of the Jacobian \( J \) is accomplished through automatic differentiation. Since the simulation is written in Fortran 2003, this is implemented using operator overloads on a dual number derived data-type which uses the chain rule to track derivatives of calculations made. Automatic differentiation has applications in several areas, including the evaluation of Jacobian matrices for Newton iteration [14, 15]. The system of equations that result from (13) are tridiagonal in form and are solved using a version of the Thomas algorithm.

During simulation runs, the spatial and temporal discretizations were refined until the solution no longer changed appreciably.

3.2. Curve fitting

A curve fit is used to match the viscosity and magnetic force constant needed to duplicate the results from experiment. This is done by first fitting a tension spline to the experimental data, which allows it to be sampled at arbitrary points in time. Next, a candidate fit is compared against this spline by defining a residual function \( R (\alpha, \beta) = \sum_k \left[ (\alpha_i (t_i) - \alpha_k (t_i))^2 \right] \), where \( \alpha_i (t_i) \) is the simulation’s angle and \( \alpha_k (t_i) \) is the experiment’s angle at the discrete time value \( t_i \). This residual function is then minimized using the Nelder-Mead simplex method with
the inputs limited to a variation of no more than 20% of the nominal value [16].

3.3. Sensitivity studies

Since previous studies of the uncertainty in oscillating cup viscometer measurements noted a linear relationship between the uncertainty of input parameters and the resultant uncertainty of the viscosity, a full Monte Carlo simulation was deemed unnecessary for the present work [17]. Since the experimental data from the only RMF viscometer experiment is available for use, the parameters of that physical experiment were used as the baseline values for the simulations, and deviations from this baseline were simulated.

The effects of each input parameter were first considered by trying to fit the experimental data using the numerical simulations, but with the parameter of interest perturbed by a relative value. This potentially resulted in altered fit results for the sample viscosity and or body force constant. The relationship between the input error and the output error (assuming the experimental data to be correct) is the subject of the present study.

In addition to single-parameter studies, certain combinations of input parameters were systematically varied simultaneously and independently to look for effects on the results from variation in two parameters. While initial efforts included a full parameter space exploration, the results showed that only two input parameters must be simultaneously considered to capture all effects of interest; this fact, combined with the computational cost of full parameter space exploration, resulted in only two parameter comparisons being made.

4. Result and analysis

To ensure that the assumed values for the simulations are relevant, the design parameters from viscometer in the literature were selected for this work. The experimental parameters of the measurement under analysis can be found in table 1.

![Table 1: Experimental parameters and conditions.](image)

**Table 1.** Experimental parameters and conditions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>$Hg$</td>
<td>Mercury</td>
<td>—</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T$</td>
<td>377</td>
<td>K</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$13.384 \times 10^3$</td>
<td>kg·m$^{-3}$</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\nu$</td>
<td>$92.1 \times 10^{-9}$</td>
<td>m$^2$·s$^{-1}$</td>
</tr>
<tr>
<td>Body force constant</td>
<td>$f_0$</td>
<td>$-4.683 \times 10^{-4}$</td>
<td>N·kg$^{-1}$</td>
</tr>
<tr>
<td>RMF Frequency</td>
<td>$\omega$</td>
<td>377.0</td>
<td>rad·s$^{-1}$</td>
</tr>
<tr>
<td>Moment of inertia</td>
<td>$I$</td>
<td>$1.867 \times 10^{-5}$</td>
<td>kg·m$^2$</td>
</tr>
<tr>
<td>Damping constant</td>
<td>$C$</td>
<td>$2.494 \times 10^{-8}$</td>
<td>N·m·s</td>
</tr>
<tr>
<td>Spring constant</td>
<td>$k$</td>
<td>$2.888 \times 10^{-6}$</td>
<td>N·m</td>
</tr>
<tr>
<td>Height of ampoule</td>
<td>$h$</td>
<td>$53.9 \times 10^{-3}$</td>
<td>m</td>
</tr>
<tr>
<td>Radius of ampoule</td>
<td>$R$</td>
<td>$4.557 \times 10^{-3}$</td>
<td>m</td>
</tr>
</tbody>
</table>

A comparison of the simulation and experimental data is shown in figure 3.

The simulation is begun right at the moment when the RMF is switched on, since prior to this the system is at rest. The transient response of the viscometer can be clearly seen as it adjusts motion of the sample induced by the magnetic field, and the momentum gained from the transition starts an oscillation of the system similar to conventional viscometer operation.

The results of the sensitivity analysis for both the sample viscosity and body force constant versus all the considered input parameters can be found in figure 4. The most noticeable characteristic of the sensitivity results is the noise in the results, which is caused by regression error in the curve fitting process. While this could be cleaned up by tightening the convergence criteria for the curve fitting process and smoothing neighboring points, leaving the noise present in the results presents a better representation of the errors as they are likely to occur in an actual measurement.

Also easily observed is the regression failure present in the results when the moment of inertia ($I$) is too far off the actual value. This is caused by the change in oscillation period caused by variation in $I$ which cannot be accounted for by other variables, especially the viscosity and body force constant, and has been observed in previous analysis [17]. This problem can be remedied by allowing $I$ to vary during the fitting process and then verifying that the fitted moment of inertia matches the measured value afterwards. Otherwise, it can be seen from the data that it must be known to within 5% to obtain a fitted results, and that even within that range any error in $I$ will strongly affect the computed viscosity. The effects from error in $I$ on $f_0$ are much less pronounced than for viscosity, since the magnetically induced fluid velocity has the strongest influence on the angular offset caused by the motion, and not the oscillations which the moment of inertia has such a profound effect on.

![Figure 3. Comparison of simulation with experimental data.](image)
outer radius ($R$) and the damping constant of the suspension thread ($C$). While it may surprise no one that the vessel radius can cancel out all of the calculations, the insignificance of the damping constant is interesting and specific to the RMF viscometer operation. In the traditional oscillating cup viscometer method, the logarithmic decrement from which the viscosity is computed includes effects from both the viscous fluid flow and the hysteretic damping within the suspension thread; the latter value must be known to compute the former, as the two affect the system oscillation in roughly the same manner. For this property to be unimportant to the RMF-based viscometer represents a significant advantage and must be a consequence of the measurement’s reliance on transient behavior instead of semi-steady state operation.

The error caused in the body force constant seems to directly relate to errors in the thread spring stiffness. This can be explained by the fact that both the body force constant and spring constant have a strong affect on the offset angle at which the system oscillates after the RMF is activated. An increase in spring constant will decrease this angle as the same viscous drag from the sample results in less displacement, while an increase in body force constant will increase the displacement as a greater fluid velocity, and thereby, viscous drag is induced. These effects are readily seen in the figure as $f_0$ decreases to compensate for increases to $k$. Nonlinearity in the momentum conservation equation causes the nonlinear relationship between $f_0$ and $k$.

It is interesting to note the similarity of the error in viscosity due to the moment of inertia and spring constants, where both show a roughly parabolic trend offset from the correct value. The similar behavior of the two parameters can be explained by their close relation in the natural frequency of an undamped system, where $f_0 \propto \sqrt{k/I}$. Thus, error in either $I$ or $k$ will result in errors in the predicted oscillation frequency of the system, and therefore, to bad fit. The position of $k$ in the numerator of the fraction is likely the reason that moderate (~10%) errors in $k$ do not cause regression failure as they do with errors in $I$. This relationship between $k$ and $I$ was further explored by considering the error in both viscosity and body force constant as a function of spring stiffness and moment of inertial simultaneously. The results of this exploration can be seen in figure 5.

The search for correlation between $k$ and $I$ in figure 5(a) proved to be quite fruitful, as the natural frequency explanation

![Figure 4](image1.png) ![Figure 5](image2.png)
of the characteristics in the single variable sensitivity study are confirmed by the new data. It can be clearly seen in the figure that the computed value for the viscosity remains largely unchanged so long as the ratio \( k/I \) is held constant as demonstrated by the 45° angle visible in the equally-scaled relative errors. Even more convincing is the absence of such a trend in the computed body force constant in figure 5(b). While the spring stiffness influence on \( f_0 \) is fairly strong, so long as regression error does not occur, \( I \) has little effect on its value; these trends are clearly visible in the data. Also of note are the regions of regression error visible in both plots of figure 5 where the ratio \( k/I \) deviates too greatly from its correct value.

The vessel height seemingly has little effect on the computed viscosity, while the relationship to the computed body force constant appears to be linear as seen in figure 4(b). An increase in \( h \) will directly increase the viscous force acting on the viscometer from the fluid, but will not affect the moment of inertia, since the two are assumed to be measured separately. While this results in an increased effect on the mean offset angle with the RMF active, it does not significantly affect the effective damping cause by the fluid on the viscometer. Upon further investigation, the moment of inertia was found to experience a decreased range of regression success with decreased values of \( h \), as can be seen in figure 6.

The derivatives of both the body force constant and viscosity as computed in the sensitivity studies are found in table 2. Assuming a roughly linear relationship between input and output, which seems reasonable given the data, the expected measurement uncertainties can be computed using the Taylor’s series method of uncertainty propagation without the need for a full Monte Carlo simulation.

From the computed sensitivities and assuming that all input parameters are known to within 1% using Gaussian statistics, the expected uncertainties of the new measurement are 1.25% for the body force constant and 3.5% for the viscosity. It must be stressed that these values assume a particular input uncertainty which is, in fact, unknown. These uncertainty results are similar to those found in the literature for the traditional oscillating-cup viscometer design, suggesting that while the RMF viscometer demonstrates numerous advantages, improved precision is likely not one of them.

5. Conclusions

From the studies conducted, it must be concluded that the error modes in the RMF-based viscometer measurement are quite different from the logarithmic decrement method traditionally used. While the moment of inertia for the viscometer must be accurately known for both methods, other parameters which have great sway on the accuracy of the traditional technique don't matter at all for the RMF variant. These benefits alone would warrant further study, but when combined with the possibility to measure the electrical conductivity and viscosity so rapidly that their variation with temperature could be reasonably developed demonstrates the new method’s dramatic superiority for electrically conducting liquids. Given the advantages, even though the resulting viscosity values are likely to be only as precise as the results of the more traditional measurement, the new method is clearly superior to the old. More careful considerations of the viscometer design, as well as even simulated experiments with more advantageous geometries should be conducted.

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