Continuous Polynomial Root Dragging

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Thank you!

I hope we have fun!
My plan

- Moving roots discretely
My plan

- Moving roots discretely
- Moving roots continuously
My plan

- Moving roots discretely
- Moving roots continuously
- Ugly formulas
My plan

- Moving roots discretely
- Moving roots continuously
- Ugly formulas
- One really cool formula
Let $p(x)$ be a polynomial of degree $n$. If $p(x)$ has $n$ real roots $r_1 \leq r_2 \leq \cdots \leq r_n$, then $p(x)$ has $n - 1$ real critical points, $r_k \leq c_k \leq r_{k+1}$. 
Theorem 1 (PRDT, [Pey67], [And93]). Suppose $p(x)$ is a polynomial of degree $n$ with $n$ real roots $r_1, \ldots, r_n$. When we drag a subset of the roots to the right, the critical points either stay where they are or move to the right.
Theorem 1 (PRDT, [Pey67], [And93]). Suppose $p(x)$ is a polynomial of degree $n$ with $n$ real roots $r_1, \ldots, r_n$. When we drag a subset of the roots to the right, the critical points either stay where they are or move to the right.

In fact, they all move, except if $r_i = c_i = r_j$, $i \neq j$, and neither $r_i$ nor $r_j$ gets moved.
Polynomial Root Dragging Theorem
What if some roots move to the right, while others move to the left?
To study this, introduce a time parameter:

\[ r_i = a_i + v_i t \]

Then let \( c(t) \) denote a critical point of

\[ p_t(x) = \prod_{k=1}^{n} (x - r_i). \]
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What is the sign of \( c'(t) \)?
Here’s an animation of the graph of

\[ p_t(x) = \prod_{k=1}^{3} (x - r_i), \text{ where } (a_1, a_2, a_3) = (-2, 0, 1) \]

and \((v_1, v_2, v_3) = (2, 1, -2)\).
Example. For $n = 3$, $p_t(x)$ is

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - (r_1 r_2 r_3).$$
Elementary symmetric functions

**Example.** For $n = 3$, $p_t(x)$ is

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$$p_t(x) = \sum_{k=0}^{n} (-1)^k \sigma_k x^{n-k},$$

$$\sigma_k = \sum_{A \subseteq \{1 \ldots n\}} \left( \prod_{i \in A} r_i \right) \text{, } |A| = k$$
In search of $c'(t)$

Notice that \( \frac{\partial}{\partial r_2}(r_1r_2 + r_1r_3 + r_2r_3) = r_1 + r_3 \), and
\[
\frac{\partial}{\partial r_2}(r_1r_2r_3) = r_1r_3.
\]
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and
\[ \frac{\partial}{\partial r_2} (r_1 r_2 r_3) = r_1 r_3. \]

In general, \( \frac{\partial}{\partial r_i} \sigma_{j+1} = \sigma^i_j \), with
\[ \sigma^i_j = \sigma_j - r_i \sigma^i_{j-1}. \]
In search of $c'(t)$

Notice that $\frac{\partial}{\partial r_2}(r_1 r_2 + r_1 r_3 + r_2 r_3) = r_1 + r_3$, and $\frac{\partial}{\partial r_2}(r_1 r_2 r_3) = r_1 r_3$.

In general, $\frac{\partial}{\partial r_i} \sigma_{j+1} = \hat{\sigma}_j^i$, with

$$\sigma_j^i = \sigma_j - r_i \sigma_{j-1}^i.$$

e.g., $r_1 r_3 = (r_1 r_2 + r_1 r_3 + r_2 r_3) - r_2(r_1 + r_3)$. 
In search of $c'(t)$

We have set $\frac{dr_i}{dt} = v_i$. We can apply implicit differentiation to obtain:

$$\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[ \left( (-1)^k (n - k - 1) c^{n-k-2} \right) \sum_{i=1}^{n} v_i \sigma_{\hat{i}} \right]}{\sum_{k=0}^{n-2} (-1)^k (n - k)(n - k - 1) c^{n-k-2} \sigma_k}.$$
In search of $c'(t)$

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$$\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[ ((-1)^k(n - k - 1)c^{n-k-2}) \sum_{i=1}^{n} v_i \hat{\sigma}_k \right]}{\sum_{k=0}^{n-2} (-1)^k(n - k)(n - k - 1)c^{n-k-2} \sigma_k}.$$ 

**Example.** If $v_i = 1$ for every $i$, this is

$$\frac{\sum_{k=0}^{n-2} \left[ \sum_{i=1}^{n} \sigma_k \right]}{\sum_{k=0}^{n-2} (n - k) \sigma_k} = 1$$

as we ought to expect.
Special case: $c(t_0) = 0$

If we assume $c(t_0) = 0$, that gets much nicer!

\[
\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[ ((-1)^k(n - k - 1)c^{n-k-2}) \sum_{i=1}^{n} v_i \sigma_{i}^k \right]}{\sum_{k=0}^{n-2} (-1)^k(n - k)(n - k - 1)c^{n-k-2}\sigma_k}
\]

\[
= \sum_{i=1}^{n} \frac{\sigma_{i}^n}{2\sigma_{n-2}} v_i.
\]
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\frac{dc}{dt} = \frac{\sum_{k=0}^{n-2} \left[ \left( (-1)^k (n - k - 1) c^{n-k-2} \right) \sum_{i=1}^{n} v_i \sigma^i_k \right]}{\sum_{k=0}^{n-2} (-1)^k (n - k)(n - k - 1) c^{n-k-2} \sigma_k}
$$

$$
= \sum_{i=1}^{n} \frac{\sigma^i_{n-2}}{2\sigma_{n-2}} v_i.
$$

We need to simplify the $\sigma^i_{n-2}$. 
Eliminating $\hat{\sigma}_n^{i-2}$

Inductively, rewrite $\sigma_j^i = \sigma_j - r_i \sigma_{j-1}^i$. 
Eliminating $\sigma_{n-2}^i$

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We get: $\sigma_{n-2}^i = \sum_{k=0}^{n-2} (-1)^{n-k} \sigma_k r_i^{n-k}$. 
Eliminating $\hat{\sigma}_{n-2}$

We get: $\hat{\sigma}_{n-2} = \sum_{k=0}^{n-2} (-1)^{n-k} \sigma_k r_i^{n-k}$.

Recalling $p_t(x) = \sum_{k=0}^{n} (-1)^k \sigma_k x^{n-k}$,

$$\sigma_n - r_i \sigma_{n-1} + r_i^2 \hat{\sigma}_{n-2} = \sum_{k=0}^{n} (-1)^{n-k} \sigma_k r_i^{n-k}$$

$$= (-1)^n p_{t_0}(r_i)$$

$$= 0.$$
Eliminating $\hat{\sigma}_n^{n-2}$

Recalling $p_t(x) = \sum_{k=0}^{n} (-1)^k \sigma_k x^{n-k}$,

$$\sigma_n - r_i \sigma_{n-1} + r_i^2 \hat{\sigma}_{n-2} = \sum_{k=0}^{n} (-1)^{n-k} \sigma_k r_i^{n-k}$$

$$= (-1)^n p_{t_0}(r_i)$$

$$= 0.$$ 

Thus $\hat{\sigma}_{n-2} = \frac{r_i \sigma_{n-1} - \sigma_n}{r_i^2}$. 
Eliminating $\sigma_{\hat{i}}$ of $n-2$

$$\sigma_n - r_i \sigma_{n-1} + r_i^2 \sigma_{n-2} = \sum_{k=0}^{n} (-1)^{n-k} \sigma_k r_i^{n-k}$$

$$= (-1)^n p_{t_0}(r_i)$$

$$= 0.$$ 

Thus $\sigma_{\hat{i}} = \frac{r_i \sigma_{n-1} - \sigma_n}{r_i^2}$. 

But the $\sigma_j$ are coefficients of $p_{t_0}(x)$, so

$$\sigma_n = p_{t_0}(0),$$

$$\sigma_{n-1} = p'_{t_0}(0) = 0,$$

$$\sigma_{n-2} = p''_{t_0}(0)/2.$$
Eliminating $\sigma_{n-2}$

Thus $\sigma_{n-2}^i = \frac{r_i \sigma_{n-1} - \sigma_n}{r_i^2}$.

But the $\sigma_j$ are coefficients of $p_{t_0}(x)$, so

\[
\begin{cases}
\sigma_n = p_{t_0}(0), \\
\sigma_{n-1} = p'_{t_0}(0) = 0, \\
\sigma_{n-2} = p''_{t_0}(0)/2.
\end{cases}
\]

Hence

\[
\frac{dc}{dt} = \sum_{i=1}^{n} \frac{\sigma_{n-2}}{2\sigma_{n-2}} v_i = \frac{-p_{t_0}(0)}{p''_{t_0}(0)} \sum_{i=1}^{n} \frac{v_i}{r_i^2}.
\]
**Gravity Theorem**

**Theorem 2.** *Suppose that* \( c(t) \) *is a critical point of* \( p_t(x) \) *for all* \( t \), *with* \( c \) *differentiable and* \( c(t_0) = 0 \). *Then*

\[
c'(t_0) = \frac{-p_{t_0}(0)}{p''_{t_0}(0)} \sum_{i=1}^{n} \frac{v_i}{r_i^2}.
\]

We can translate any \( p_t(x) \) to put any one of its critical points at \( x = 0 \) when \( t = 0 \). Thus we’ve shown in general that the effect of roots on critical points is analogous to the effect of gravity on masses.
Thank You!

References


